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Localised solutions of a non-linear spinor field

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Abstract. The existence and structure of localised solutions for a non-linear spinor field are studied.

1. Introduction

Recently the solutions of classical non-linear field equations have received considerable attention as a means to obtain information on the structure of the corresponding quantum field theories (Gervais and Neveu 1976). The main idea is to look for classical solutions to non-linear field equations and construct quantum states around them. In particular those solutions correspond to the expectation values of quantum fields on suitable coherent states (Hepp 1974). The JWKB method, the Feynman path integral method, etc are based on the properties of the classical solutions. Moreover classical field theory is, at least, the order-zero approximation to quantum field theory. For these reasons it is interesting to study the properties of the classical solutions of non-linear field equations.

The Dirac field with a positive $(\bar{\psi}\psi)^2$ self-interaction, introduced by Weyl (1950), has been studied numerically by Soler (1970) and Rañada *et al* (1974) who were looking for localised solutions which could be used as representations of extended particles, like nucleons. In § 2 we study some properties of those solutions. In particular we prove that they have an exponential decay at infinity. This property of the solutions is used to recognise them when they are computed numerically. Also, we prove the non-existence of localised solutions when $\Lambda^2 > 1$ (Λ is the representative parameter in the model). Thus, in order to find localised solutions, we only have to consider the values of Λ, such that $\Lambda^2 \leq 1$.

In § 3 we prove the existence of a one-parameter family of localised solutions for a non-linear scalar field obtained as a Klein–Gordon limit for the above non-linear spinor field.

2. Classical Dirac field with a $(\bar{\psi}\psi)^2$ self-coupling

The Lagrangian studied is the following:

$$\mathscr{L}_{\mathrm{D}} = \tfrac{1}{2}\mathrm{i}[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi] - M\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^2. \tag{1}$$

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Our notation will be

$$g_{\mu\mu}=(1,-1,-1,-1) \qquad \gamma^0=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^\kappa=\begin{pmatrix} 0 & \sigma^\kappa \\ \sigma^\kappa & 0 \end{pmatrix} \tag{2}$$

where σ^κ are the Pauli matrices and λ is a positive constant.

The field equations are

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - M\psi + 2\lambda(\bar{\psi}\psi)\psi = 0 \tag{3}$$

and they admit the following stationary solutions which are separable in spherical coordinates:

$$\psi(\boldsymbol{x}, t) = \mathrm{e}^{-\mathrm{i}\omega t}\begin{pmatrix} g(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathrm{i}f(r)\begin{pmatrix} \cos\theta \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} \end{pmatrix}. \tag{4}$$

The following convenient transformations in functions and variables are made:

$$(f, g)=\left(\frac{M}{2\lambda}\right)^{1/2}(F, G),$$

$$\rho = Mr, \qquad \Lambda=\omega/M, \qquad \beta^2=1-\Lambda^2. \tag{5}$$

The field equations (3) are now

$$F'+2\frac{F}{\rho}+(1-\Lambda)G-(G^2-F^2)G=0 \tag{6a}$$

$$G'+(1+\Lambda)F-(G^2-F^2)F=0 \tag{6b}$$

the localised solutions of (6) are solutions $F(|\boldsymbol{x}|)$, $G(|\boldsymbol{x}|)$ which are functions of class $C^2(\mathbb{R}^3)$ which are bounded. They belong to the Sobolev space $H^1(\mathbb{R}^3)$ (the space of L^2 functions whose generalised first-order derivatives belong to L^2) such that

$$\lim_{\rho\to\infty} \rho^{3/2}\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $F(0)=0$. It is possible to compute them numerically, but to prove their existence and to characterise all of them are open questions. It is difficult to apply the variational methods of Berger (1972) and Ambrosetti and Rabinowitz (1973) because: (i) the localised solutions of (6) are the critical points of an even functional $I(F, G)$ subject to a constraint which is not positive definite and such that $I(F, G)=0$ if $F=G\neq 0$ (lemma 2(*a*)); (ii) the localised solution of (6) satisfies equations (8) which correspond to two coupled Klein–Gordon fields, with two self-couplings of fourth and sixth order, and in this case it is not possible to apply Ljusternik's theorems since the functional $\int_{\mathbb{R}^3} u^6\, \mathrm{d}^3x$ is not completely continuous in $H^1(\mathbb{R}^3)$.

Lemma 1. If $\Lambda^2>1$ there is no localised solution for equations (6).

Proof. Applying the operator $\mathrm{i}\gamma^\partial\partial_\nu$ to (3) we get an equation like the Klein–Gordon equation:

$$\Delta\psi+(\omega^2-M^2)\psi+4\lambda M(\bar{\psi}\psi)\psi-4\lambda^2(\bar{\psi}\psi)^2\psi+2\mathrm{i}\lambda[\gamma^\alpha\partial_\alpha(\bar{\psi}\psi)]\psi=0. \tag{7}$$

This transformation is right because we work with radial functions in $H^1(\mathbb{R}^3)$ and their L^p-norms $(2 \leq p \leq 6)$ are bounded by their norm in $H^1(\mathbb{R}^3)$ (Nirenberg 1959).

Using the transformations (5) and equations (6) we have the radial equations:

$$F'' + \frac{2}{\rho}F' - \frac{2}{\rho^2}F - \beta^2 F + 2(G^2 - F^2)F - (G^2 - F^2)^2 F + \left(4\Lambda G^2 - \frac{4FG}{\rho}\right)F = 0 \tag{8a}$$

$$G'' + \frac{2}{\rho}G' - \beta^2 G + 2(G^2 - F^2)G - (G^2 - F^2)^2 G + 4\Lambda F^2 G - \frac{4F^3}{\rho} = 0. \tag{8b}$$

We can regard (8a) as the equation

$$\Delta u + (-\beta^2 + p(\rho))u = 0 \tag{9}$$

where $u = F(\rho)\cos\theta$

$$p(\rho) = 2(G^2 - F^2) - (G^2 - F^2)^2 + \left(4\Lambda G^2 - \frac{4FG}{\rho}\right)$$

for the localised solutions $\lim_{\rho \to \infty} \rho p(\rho) = 0$, thus applying the result of Kato (1959) (9) has only the trivial localised solution $u = 0 (\Rightarrow F = 0)$ if $-\beta^2 > 0$, i.e. $\Lambda^2 > 1$. In the same way equation (8b) with $F = 0$ and $\Lambda^2 > 1$ has only the trivial localised solution $G = 0$.

So we conclude that if $\Lambda^2 > 1$ there are no localised solutions for the equations (6).

Lemma 2. The localised solutions of (6) satisfy the following properties.

(a) They are the critical points of the functional $I(F, G) = \int_0^\infty (G^2 - F^2)^2 \rho^2 \, d\rho$ subject to the constraint

$$\int_0^\infty \left(\Lambda(G^2 + F^2) - (G^2 - F^2) - \frac{2FG}{\rho} - GF' + FG'\right)\rho^2 \, d\rho = -R \qquad (R > 0).$$

(b) They satisfy the integral conditions

$$\int_0^\infty (G^2 + F^2)\rho \, d\rho = \int_0^\infty 2\Lambda FG\rho^2 \, d\rho \tag{10a}$$

$$\int_0^\infty (G^2 - F^2)^2 \rho^2 \, d\rho = \int_0^\infty [(\Lambda - 1)G^2 + (1 + \Lambda)F^2]\rho^2 \, d\rho \tag{10b}$$

$$\tfrac{1}{4}G^4(0) - \tfrac{1}{2}(1 - \Lambda)G^2(0) = \int_0^\infty 2(F^2/\rho)[(1 + \Lambda) + (F^2 - G^2)] \, d\rho. \tag{10c}$$

(c) They have an exponential decay at infinity.

Proof. (a) If $(\bar{G}, \bar{F})$ is a critical point of the variational problem, then

$$\bar{F}' + 2\frac{\bar{F}}{\rho} + (1 - \Lambda)\bar{G} - 2R\bar{G}(\bar{G}^2 - \bar{F}^2) = 0$$

$$\bar{G}' + (1 + \Lambda)\bar{F} - 2k\bar{F}(\bar{G}^2 - \bar{F}^2) = 0$$

where k must be a positive constant in order to satisfy the constraint. Thus $(F, G) = (2k)^{1/2}(\bar{F}, \bar{G})$ satisfies (6). As we can see if $F = G \neq 0$ then $I(F, G) = 0$ and the constraint is not positive definite.

(*b*) Equation (10*a*) is obtained by integrating directly in equations (6). This relation implies the non-existence of localised solutions when $\Lambda = 0$ and gives us a measure of the size of the localised solution:

$$\langle\rho\rangle = \frac{\int_0^\infty (F^2+G^2)\rho^2\,\mathrm{d}\rho}{\int_0^\infty (F^2+G^2)\rho\,\mathrm{d}\rho} \geqslant \frac{1}{\Lambda}.$$

To obtain (10*b*) we use an argument of Rosen (1969). Equations (6) are associated with a variational principle, in particular their corresponding Lagrangian is

$$L = \int\left[\Lambda(F^2+G^2)-(G^2-F^2)+FG'-GF'-\frac{2FG}{\rho}+\frac{1}{2}(G^2-F^2)^2\right]\rho^2\,\mathrm{d}\rho.$$

Then we have the global condition

$$\left(\frac{\mathrm{d}L}{\mathrm{d}\lambda}(F(\lambda\rho), G(\lambda\rho))\right)_{\lambda=1} = 0. \tag{11}$$

Invoking equations (6), we eliminate the derivatives in F and G, getting relation (10*b*) which implies the non-existence of localised solutions if $\Lambda \leqslant -1$.

Relation (10*c*) is obtained from the following phase-space analysis: the differential equations (6) describe a non-conservative, one-dimensional motion since ρ ('time') appears explicitly. The energy for the corresponding conservative motion (defined by equations (6) after the $1/\rho$ term has been deleted) is

$$K = \tfrac{1}{2}(1+\Lambda)F^2 - \tfrac{1}{2}(1-\Lambda)G^2 + \tfrac{1}{4}(F^2-G^2)^2$$

and for the non-conservative motion which corresponds to our actual problem we have

$$\frac{\mathrm{d}K}{\mathrm{d}\rho} = -\frac{2F^2}{\rho}[(1+\Lambda)+(F^2-G^2)]$$

since for a localised solution $F(0)=0$, $G(0)\neq 0$ and $F(\infty)=G(\infty)=0$ we thus obtain (10*c*) by integrating $\mathrm{d}K/\mathrm{d}\rho$.

The integral conditions (10) can be used in order to prove the non-existence of localised solutions, to test the accuracy of the numerical localised solutions, and to find variational solutions.

(*c*) The Green function $g_0(\rho, s)$ for the operator

$$L_0 G = G_{\rho\rho} + \frac{2}{\rho}G_\rho - \beta^2 G$$

can be written

$$g_0(\rho, s) = \begin{cases} -\dfrac{s^2}{\beta}\dfrac{\mathrm{e}^{-\beta\rho}}{\rho}\dfrac{\sinh(\beta s)}{s} & \rho \geqslant s \\[2ex] -\dfrac{s^2}{\beta}\dfrac{\mathrm{e}^{-\beta s}}{s}\dfrac{\sinh(\beta\rho)}{\rho} & \rho \leqslant s. \end{cases}$$

Thus the solution $G(\rho)$ of equation (8*b*) can be estimated:

$$|G(\rho)| \leqslant \frac{\mathrm{e}^{-\rho\beta}}{\rho\beta}\int_0^\infty \frac{\sinh(\beta s)}{s}s^2|T(s)|\,\mathrm{d}s + \frac{\sinh(\beta\rho)}{\beta\rho}\int_0^\infty \frac{\mathrm{e}^{-\rho\beta}}{s}s^2|T(s)|\,\mathrm{d}s$$

where

$$T(s) = -2G(G^2 - F^2) + (G^2 - F^2)^2 G - 4\left(\Lambda FG - \frac{F^2}{s}\right)F$$

since $\sinh(\beta\rho)/\rho$ increases and $e^{-\beta\rho}/\rho$ decreases as ρ increases, we obtain

$$|G(\rho)| \leq \frac{e^{-\rho\beta}}{\beta\rho}\frac{\sinh(\beta\rho)}{\rho}\int_0^\infty s^2|T(s)|\,ds$$

since G and F belong to $H^1(\mathbb{R}^3)$, by the Hölder and Sobolev inequalities

$$\int_0^\infty s^2|T(s)|\,ds = D_0 < \infty$$

(in particular

$$\int_0^\infty \frac{|F^3|}{\rho}\rho^2\,d\rho$$

$$\leq \frac{1}{2}\int_0^\infty F^2\,d\rho + \frac{1}{2}\int_0^\infty F^2\rho^2\,d\rho$$

$$\leq 2\int_0^\infty F'^2\rho^2\,d\rho + \frac{1}{2}\int_0^\infty F^2\rho^2\,d\rho \leq 2\|F\|^2_{H^1(\mathbb{R}^3)}\Big)$$

where D_0 depends on the norms of G and F in $H^1(\mathbb{R}^3)$. So we get

$$|G(\rho)| \leq \frac{D_0}{2\beta\rho^2} \qquad \text{as } \rho \to \infty. \tag{12}$$

In the same way we obtain for F a similar *a priori* bound. The Green function $g_1(\rho, s)$ for the operator

$$L_1 F = F_{\rho\rho} + \frac{2}{\rho}F_\rho - \frac{2}{\rho^2}F - \beta^2 F$$

can be written

$$g_1(\rho, s) = \begin{cases} \left(\dfrac{1}{\rho^2} + \dfrac{\beta}{\rho}\right) e^{-\beta\rho}\left(\dfrac{\sinh(\beta s)}{\beta^2 s^2} - \dfrac{\cosh(\beta s)}{\beta^2 s}\right)s^2 & \rho \geq s \\[2ex] \left(\dfrac{\sinh(\beta\rho)}{\beta^3\rho^2} - \dfrac{\cosh(\beta\rho)}{\beta^2\rho}\right)\left(\dfrac{1}{s^2} + \dfrac{\beta}{s}\right) e^{-\beta s}s^2 & \rho \leq s \end{cases}$$

and as before we get

$$|F(\rho)| \leq \frac{4}{\beta}\frac{D_1}{\rho^2} \qquad \text{as } \rho \to \infty \tag{13}$$

where

$$D_1 = \int_0^\infty s^2\left|-2F(G^2 - F^2) + (G^2 - F^2)^2 F - 4G\left(\Lambda FG - \frac{F^2}{s}\right)\right|\,ds$$

which depends on the norms of G and F in $H^1(\mathbb{R}^3)$.

In equation (9) (the same as (8a)) with the *a priori* bounds (12)–(13) we can regard $u(\rho)$ as an eigenvector associated with the eigenvalue $-\beta^2$, where $p(\rho) = O(1/\rho^4)$ as

$\rho \to \infty$, and is bounded as $\rho \to 0$. Hence, for some δ, $F(\rho) = O(e^{-\delta\rho})$ as $\rho \to \infty$. This follows from standard estimates from the spectral theory of second-order operators (Glazman 1965). In the same way we can regard $G(\rho)$ (equation (8*b*)) as the solution of the equation

$$\Delta G - q(\rho)G - \beta^2 G = t(\rho)$$

where

$$q(\rho) = -2(G^2 - F^2) + (G^2 - F^2)^2 - 4\Lambda F^2$$

$$t(\rho) = 4F^3/\rho.$$

Then $G(\rho)$ is the sum of two functions with exponential decay as $\rho \to \infty$. The first one corresponds to the homogeneous equation for which the above argument also holds.The other is a solution for the complete equation (12) and its exponential decay follows since $t(\rho) = O(e^{-3\delta}/\rho)$.

3. Non-linear Klein-Gordon equation

Considering the upper spinor components in equation (7) and with a suitable change in dimensions we obtain the Klein–Gordon limit

$$-\Delta\phi + (M^2 - \omega^2)\phi - \alpha(\phi^*\phi)\phi + \mu(\phi^*\phi)^2\phi = 0 \qquad (14)$$

where $\phi = e^{-i\omega t}\phi(\boldsymbol{x})$ and $\alpha = 4\lambda M^2$, $\mu = 4\lambda^2 M^2$. That corresponds to a complex scalar field with two self-couplings of fourth and sixth order, and their spherically symmetric localised solutions have been studied numerically by Anderson (1971).

Making the transformations

$$\phi = \left(\frac{M^2 - \omega^2}{\mu}\right)^{1/4} u, \qquad r = \rho(M^2 - \omega^2)^{1/2}, \qquad \epsilon = \alpha[\mu(M^2 - \omega^2)]^{-1/2}$$

we obtain

$$-\Delta u + u - \epsilon u^3 + u^5 = 0. \qquad (15)$$

A necessary condition for the existence of localised solutions is $\epsilon > (\frac{16}{3})^2$. Also, using the argument of Rosen (1969), we obtain for those solutions

$$\int u^4 \, d^3\rho = 4 \int u^2 \, d^3\rho.$$

It is possible to prove the existence of a solution of (15) for some values of ϵ. This result is a consequence of the following theorem (Berger and Berger 1968).

Theorem. Suppose: (i) $f_1(u) \to \infty$ as $\|u\| \to \infty$; grad $f_1(u)$ is monotonic and continuous in the Hilbert space H and (grad $f_1(u)$, $(u) \neq 0$ for $u \neq 0$. Suppose also that: (ii) grad $f_2(u)$ is a completely continuous operator such that grad $f_2(0) = 0$; and (iii) (grad $f_2(u)$, $u) > 0$ for $u \neq 0$. Then for every $c > 0$ there exists a $u_c \in H$ with $F(u_c) = c$ and

$$\operatorname{grad} f_1(u_c) = \eta \operatorname{grad} f_2(u_c) \qquad (16)$$

where η is a real number, that is a one-parameter family of solutions u_c of (16), each solution being a critical point of $f_2(u)$ on the manifold $\partial\Sigma_c = \{u | f_1(u) = c\}$.

To demonstrate the applicability of the above theorem we set:

$$H = H_r^1(\mathbb{R}^3) = \left\{ u(|\boldsymbol{x}|) \in L^2(\mathbb{R}^3) \middle| \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3),\ i = 1, 2, 3 \right\}$$

$$[u, v]_H = \int_{\mathbb{R}^3} (uv + \nabla u \cdot \nabla v)\, \mathrm{d}^3 r$$

$$f_1(u) = \int_{\mathbb{R}^3} [\tfrac{1}{2}(|\nabla u|^2 + u^2) + \tfrac{1}{6} u^6]\, \mathrm{d}^3 r$$

$$f_2(u) = \tfrac{1}{4} \int_{\mathbb{R}^3} u^4\, \mathrm{d}^3 r$$

(i) Clearly $f_1(u) \to \infty$ as $\|u\| = (\int_{\mathbb{R}^3} (u^2 + |\nabla u|^2)\, \mathrm{d}^3 r)^{1/2} \to \infty$ and $(\operatorname{grad} f_1(u), u) \neq 0$ for $u \neq 0$. $\operatorname{grad} f_1(u)$ is monotone:

$$(\operatorname{grad} f_1(u) - \operatorname{grad} f_1(v),\ u - v)$$
$$= \int (\nabla(u - v) \cdot \nabla(u - v) + (u - v)^2 + u^6 + v^6 - u^5 v - v^5 u)\, \mathrm{d}^3 r \geqslant 0$$

because, using the Hölder inequality and the fact that $a, b > 0$ and $0 < \alpha < 1$, $a^\alpha b^{\alpha - 1} \leqslant \alpha a + (1 - \alpha) b$, we obtain

$$\int (u^5 v + v^5 u)\, \mathrm{d}^3 r \leqslant \int (u^6 + v^6)\, \mathrm{d}^3 r.$$

Also $\operatorname{grad} f_1(u)$ is continuous:

$$(\operatorname{grad} f_1(u), v) \leqslant \tfrac{1}{2}(\|u\|^2 + \|v\|^2) + \|u\|^5 \|v\|$$

since for all $u \in H^1(\mathbb{R}^3)$

$$\int u^6\, \mathrm{d}^3 r \leqslant K_6 \left(\int (|\nabla u|^2 + u^2)\, \mathrm{d}^3 r \right)^{1/3}$$

where K_6 is a constant independent of u.

(ii) $\operatorname{grad} f_2(0) = 0$ and $(\operatorname{grad} f_2(u), u) > 0$ for $u \neq 0$ and $\operatorname{grad} f_2(u)$ is completely continuous (Berger 1972).

Thus the hypotheses of the theorem are satisfied. The equation associated with the variational problem is (15) with $\eta = \epsilon$ and this has a one-parameter u_c family of weak solutions. By the classical regularity theory of critical points (due to Hilbert and Tonelli) each u_c is smooth enough to satisfy (15) and the point-wise sense (Berger 1972).

The theorem does not tell us how to compute the values $\eta (= \epsilon)$.

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References

Ambrosetti A and Rabinowitz P H 1973 *J. Funct. Anal.* **14** 349–81
Anderson D L T 1971 *J. Math. Phys.* **12** 945–52
Berger M 1972 *J. Funct. Anal.* **9** 249–61
Berger M and Berger M 1968 *Perspectives in Nonlinearity* (New York: Benjamin) pp 113–6
Gervais J L and Neveu A 1976 *Phys. Rep.* **23** 240–374
Glazman I 1965 *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Jerusalem: Israel Program for Scientific Translations)
Hepp K 1974 *Commun. Math. Phys.* **35** 265–77
Kato T 1959 *Commun. Pure Appl. Math.* **12** 403–25
Nirenberg L 1959 *Ann. Scuola Norm. Sup. Pisa* **13** 115–62
Rañada A F, Rañada M F, Soler M and Vazquez L 1974 *Phys. Rev.* D **10** 517–25
Rosen G 1969 *Q. Appl. Math.* **27** 133–4
Soler M 1970 *Phys. Rev.* D **1** 2766–9
Weyl H 1950 *Phys. Rev.* **77** 699–701